

A mild source for the Wu-Yang magnetic monopole

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Abstract

We establish that the Wu-Yang monopole needs the introduction of a magnetic point source at the origin in order for it to be a solution of the integral equations for the Yang-Mills theory. That result is corroborated by the analysis of the differential Yang-Mills equations using distribution theory. The subtlety lies on the fact that with the non-vanishing magnetic point source required by the Yang-Mills integral equations, the Wu-Yang monopole configuration does not violate, in the sense of distribution theory, the differential Bianchi identity.

The purpose of this paper is to settle a long standing problem concerning the nature of the singularity of the magnetic monopole solution constructed in 1969 by T.T. Wu and C.N. Yang [1] for the pure $SU(2)$ Yang-Mills differential field equations. The solution presents a spherically symmetric non-abelian magnetic field with a strength that depends on the inverse of the square of the radial distance, for all distance scales, and so it is singular at the origin. Such singularity has been an issue since then. Not only those authors pointed out that [2] but many others [3, 4, 5] have discussed this problem, in particular the question of whether a point source is needed to sustain the configuration. In this paper we discuss how the integral Yang Mills equations, proposed in [6, 7], shed a light on this question by revealing that the concerns presented in [2] with respect to the compatibility of the Wu-Yang configuration with the Bianchi identity are indeed very strong and we show how those integral equations establish that the introduction of a point source (uniquely fixed by them) indeed solves this long standing question. The crucial and interesting aspect is that with such non-vanishing magnetic point source required by the Yang-Mills integral equations, the Wu-Yang monopole configuration does not violate, in the sense of distribution theory, the differential Bianchi identity.

Another intriguing point concerning the Wu-Yang monopole solution is that so far it did not really possess a magnetic charge associated to it. Indeed, being a solution of the pure Yang-Mills theory without a Higgs field, it does not possess, like the 't Hooft-Polyakov monopole does [8, 9], a topological charge that can be interpreted as a magnetic charge. In addition, the usual (dynamically conserved) Noether magnetic charge of Yang-Mills theories vanishes when evaluated on the Wu-Yang monopole solution. We show in this paper that the Wu-Yang monopole does possess a dynamically conserved non-vanishing magnetic charge. It is constructed through the integral equations for the Yang-Mills theory [6, 7], and its conservation comes from an iso-spectral time evolution of some special operators, in a manner similar to what happens in integrable field theories, and so it is not a Noether charge. In addition, contrary to the usual Noether magnetic charge, such charge is invariant under general (large) gauge transformations.

In order to address the singularity issue of the Wu-Yang monopole we use the recently proposed integral equations for the Yang-Mills theory [6, 7] to fix in a unique way the type of point source one needs to introduce to make the solution consistent. The result obtained is then corroborated by the use of distribution theory in the differential Yang-Mills equations. The result we find is that the magnetic field of the Wu-Yang monopole solution must satisfy

$$D_i B^i = \frac{1}{e} \frac{\hat{r} \cdot T}{r^2} \delta(r) \quad (1)$$

where $\delta(r)/r^2$ is the radial part of the three dimensional Dirac delta function $\delta^{(3)}(\vec{r})$, e is the gauge coupling constant, $\hat{r} = \vec{r}/r$ is the unit vector in the radial direction, and $\hat{r} \cdot T = \hat{r}_i T_i$, with

T_i being the generators of the $SU(2)$ Lie algebra, i.e. $[T_i, T_j] = i \varepsilon_{ijk} T_k$, $i, j, k = 1, 2, 3$. The magnetic field is defined as $B_i = -\frac{1}{2} \varepsilon_{ijk} F_{jk}$, with the field tensor being $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i e [A_\mu, A_\nu]$, $\mu, \nu = 0, 1, 2, 3$, and the covariant derivative being $D_\mu \star = \partial_\mu \star + i e [A_\mu, \star]$. With such a notation the Wu-Yang solution [1] reads

$$A_i = -\frac{1}{e} \varepsilon_{ijk} \frac{x^j}{r^2} T_k, \quad F_{ij} = \frac{1}{e} \varepsilon_{ijk} \frac{x^k}{r^3} \hat{r} \cdot T \quad (2)$$

with $A_0 = 0$ and $F_{0i} = 0$. The fact that the point source in (1) contains only the radial part of the Dirac delta function will prove to be crucial for the compatibility between the results about the analyticity of the solution, obtained through the integral equations of Yang-Mills theory and that of distribution theory applied to the differential Yang-Mills equations. In addition, it makes the source spherically symmetric under the joint action of physical space and isospin space rotations.

The integral equations for the Yang-Mills theory were obtained in [6, 7] using a generalization [10, 11] of the non-abelian Stokes theorem [12] for a pair $(\mathcal{B}_{\mu\nu}, A_\mu)$, of an antisymmetric tensor $\mathcal{B}_{\mu\nu}$ and a one-form connection A_μ , as follows. In a space-time M consider a three-volume Ω with border $\partial\Omega$. Choose a reference point x_R on $\partial\Omega$ and scan Ω with closed surfaces, based on x_R , labelled by ζ , such that $\zeta = 0$ corresponds to the infinitesimal surface around x_R , and $\zeta = \zeta_0$ corresponds to $\partial\Omega$. Each closed surface scanning Ω , is scanned in its turn by closed loops, based on x_R , labelled by τ , such that $\tau = 0$ and $\tau = 2\pi$ correspond to the infinitesimal loops around x_R , at the beginning and ending of the scanning. Each loop on its turn is parameterized by σ , starting and ending at x_R , such that $\sigma = 0$ and $\sigma = 2\pi$ correspond to the end points of the loop. The generalized non-abelian Stokes theorem [10, 11, 6, 7] states that

$$V(\partial\Omega) \equiv P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} \mathcal{B}_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta \mathcal{K}} \equiv U(\Omega) \quad (3)$$

where P_3 and P_2 mean volume and surface ordering respectively, according to the scanning described above. The quantity V , on the left, is obtained by integrating the equation

$$\frac{dV}{d\tau} - V T(\mathcal{B}, A, \tau) = 0 \quad \text{with} \quad T(\mathcal{B}, A, \tau) \equiv \int_0^{2\pi} d\sigma W^{-1} \mathcal{B}_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \quad (4)$$

where the σ -integration is along the loop labeled by τ , and W is the Wilson line obtained by integrating, along the loop, the equation

$$\frac{dW}{d\sigma} + i e A_\mu \frac{dx^\mu}{d\sigma} W = 0. \quad (5)$$

On the other hand the quantity on the right, defined here as $U(\Omega)$, is obtained by integrating the equation

$$\frac{dU}{d\zeta} - \mathcal{K} U = 0 \quad (6)$$

where \mathcal{K} is given by

$$\begin{aligned} \mathcal{K} \equiv & \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma V \left\{ W^{-1} [D_\lambda \mathcal{B}_{\mu\nu} + D_\mu \mathcal{B}_{\nu\lambda} + D_\nu \mathcal{B}_{\lambda\mu}] W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\zeta} \right. \\ & - \int_0^\sigma d\sigma' \left[\mathcal{B}_{\kappa\rho}^W(\sigma') - ie F_{\kappa\rho}^W(\sigma'), \mathcal{B}_{\mu\nu}^W(\sigma) \right] \frac{dx^\kappa}{d\sigma'} \frac{dx^\mu}{d\sigma} \\ & \left. \times \left(\frac{dx^\rho(\sigma')}{d\tau} \frac{dx^\nu(\sigma)}{d\zeta} - \frac{dx^\rho(\sigma')}{d\zeta} \frac{dx^\nu(\sigma)}{d\tau} \right) \right\} V^{-1} \end{aligned} \quad (7)$$

where we have introduced the notation $X^W \equiv W^{-1} X W$. The integral equations for the Yang-Mills theory [6, 7] is obtained from such generalized non-abelian Stokes theorem by taking the connection A_μ to be the Yang-Mills gauge field, thus satisfying

$$D_\nu F^{\nu\mu} = J^\mu \qquad D_\nu \tilde{F}^{\nu\mu} = j^\mu \quad (8)$$

where we allowed for a magnetic current j_μ , besides the matter current J_μ , acting as a source, and the antisymmetric tensor $\mathcal{B}_{\mu\nu}$ to be a linear combination of the field tensor and its Hodge dual, i.e.

$$\mathcal{B}_{\mu\nu} = ie \left[\alpha F_{\mu\nu} + \beta \tilde{F}_{\mu\nu} \right] \quad (9)$$

where $\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$. The parameters α and β are arbitrary and can in fact be even complex.

The crucial property of the Wu-Yang solution, for our calculation, is that the field tensor conjugated by the Wilson line, namely $W^{-1} F_{ij} W$, has a fixed direction in the Lie algebra, and so effectively becomes abelian. That is a consequence of the fact that the Lie algebra element $\hat{r} \cdot T$ is covariantly constant. Indeed, one can check that for the connection (2) one has $D_i \hat{r} \cdot T = 0$. Therefore, using (5) one gets

$$\frac{d}{d\sigma} \left(W^{-1} \hat{r} \cdot T W \right) = W^{-1} (D_i \hat{r} \cdot T) W \frac{dx^i}{d\sigma} = 0. \quad (10)$$

Consequently, the quantity $W^{-1} \hat{r} \cdot T W$ is the same in any point of the loop and so equal to its value at the reference point x_R . By fixing the integration constant in (5) to be unity one gets that the Wilson line W is unity at the reference point. Therefore

$$W^{-1} \hat{r} \cdot T W = T_R; \qquad \text{and so} \qquad W^{-1} F_{ij} W = \frac{1}{e} \varepsilon_{ijk} \frac{x^k}{r^3} T_R \quad (11)$$

where $T_R \equiv \hat{r}_R \cdot T$, and \hat{r}_R is the unit radial vector at the reference point x_R , and where we have used (2).

In order to use the Yang-Mills integral equations to find the magnetic current j^μ (see (8)) such that the Wu-Yang monopole configuration (2) becomes a solution we shall consider a

purely spatial three-volume Ω , and so all derivatives of the time coordinate x^0 , w.r.t. the parameters σ , τ and ζ vanish. Therefore, the components \mathcal{B}_{0i} and A_0 , of the antisymmetric tensor and of the gauge field, will not enter in the calculation (in fact $A_0 = 0$ for the solution (2)). Since $\tilde{F}_{ij} = 0$ for the Wu-Yang monopole (2), one has, from (9), that $\mathcal{B}_{ij} = i e \alpha F_{ij}$. The integration to obtain V in (4) does not need the surface ordering because the integrand in the definition of $T(\mathcal{B}, A, \tau)$ lies in the direction of T_R for any value of σ and τ . So one has that

$$V(\partial\Omega) = \exp \left[i \alpha T_R \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma \varepsilon_{ijk} \frac{x^k}{r^3} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \right] = \exp \left[i \alpha T_R \int_{\partial\Omega} d\vec{\Sigma} \cdot \hat{r} \right] \quad (12)$$

where in the last equality we have used the fact that $\varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} d\sigma d\tau = r^2 d\Sigma_k$, with $r^2 d\vec{\Sigma}$ being the vector perpendicular to the surface $\partial\Omega$ at any given point, and whose modulus is the area element of $\partial\Omega$ at that point. We shall assume that the scanning of the surface $\partial\Omega$ by loops, parameterized by σ and τ , is such that $d\vec{\Sigma}$ points outward $\partial\Omega$. If that is not the case one gets a minus sign in the definition of $d\vec{\Sigma}$. Consequently, $d\vec{\Sigma} \cdot \hat{r}$ is the solid angle element of $\partial\Omega$ seen from the origin of the coordinate system. Therefore, one has that

$$V(\partial\Omega) = \begin{cases} \exp[i 4 \pi \alpha T_R] & \text{if the origin is inside } \Omega \\ \mathbb{1} & \text{if the origin is outside } \Omega \end{cases} \quad (13)$$

Another consequence of (11) is that the commutator term in (7) vanishes, i.e.

$$[\mathcal{B}_{ij}^W(\sigma') - i e F_{ij}^W(\sigma'), \mathcal{B}_{kl}^W(\sigma)] = -e^2 \alpha (\alpha - 1) [F_{ij}^W(\sigma'), F_{kl}^W(\sigma)] = 0 \quad (14)$$

Using (8) one gets that

$$(D_i F_{jk} + D_j F_{ki} + D_k F_{ij}) \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \frac{dx^k}{d\zeta} = j^0 \varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \frac{dx^k}{d\zeta}. \quad (15)$$

Therefore, the quantity \mathcal{K} introduced in (7), for the Wu-Yang monopole and for the case of Ω being purely spatial, becomes

$$\mathcal{K} = i e \alpha \int_0^{2\pi} \int_0^{2\pi} d\tau d\sigma V W^{-1} j^0 W V^{-1} \varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \frac{dx^k}{d\zeta}. \quad (16)$$

We need the integration of (6) to give an operator $U(\Omega)$ that equals $V(\partial\Omega)$ given in (13) (see (3)). We do not see any other way of satisfying that condition but to have j^0 to lie in the direction of $\hat{r} \cdot T$. Indeed, if one takes $j^0 = f(\vec{r}) \hat{r} \cdot T$, then one gets that $V(\tau) W^{-1} j^0 W V^{-1}(\tau) = f(\vec{r}) T_R$, where we have used (11), and the fact that $V(\tau)$ commutes with T_R , since it is obtained by integrating (4) using the fact the $T(\mathcal{B}, A, \tau)$ is proportional to T_R for the case under consideration (see (11)). The integration of (6) is now very simple and gives

$$U(\Omega) = \exp \left[i e \alpha T_R \int_0^{\zeta_0} d\zeta \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma f(\vec{r}) \varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \frac{dx^k}{d\zeta} \right]. \quad (17)$$

We have assumed that the scanning of the surfaces with loops, parameterized by σ and τ , is such that the vector $\varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau}$ points outwards the surface. Since ζ grows in the direction outwards the surface, it turns out that $d\zeta d\tau d\sigma \varepsilon_{ijk} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} \frac{dx^k}{d\zeta} = d^3\vec{r}$ (if the scanning does not satisfy that condition one gets $-d^3\vec{r}$ instead). Since $U(\Omega)$ given above has to equal (13) for any volume Ω , it turns out that $f(\vec{r})$ has to be sharply localized at the origin. However, it can not be proportional to the three-dimensional Dirac delta function because we have assumed that $j^0 = f(\vec{r}) \hat{r} \cdot T$, and so that would fix the angular dependence of $\hat{r} \cdot T$ and consequently we could not use (11), as we have done already. In addition, as we show below, the choice of $f(\vec{r})$ being proportional to the three-dimensional Dirac delta function is incompatible with the analysis of the differential Yang-Mills equation using distribution theory, as we show below. We therefore take $f(\vec{r})$ to be proportional to the radial part of the three-dimensional Dirac delta function, i.e. we take $f(\vec{r}) = \frac{1}{e} \frac{\delta(r)}{r^2}$, and then one gets from (17) that

$$U(\Omega) = \begin{cases} \exp[i 4 \pi \alpha T_R] & \text{if the origin is inside } \Omega \\ \mathbb{1} & \text{if the origin is outside } \Omega. \end{cases} \quad (18)$$

Comaparing (13) and (18) we conclude that the Wu-Yang monopole solution (2) satisfies the integral Yang-Mills equations (see (3)) if the magnetic current is given by

$$j^0 = \frac{1}{e} \frac{\delta(r)}{r^2} \hat{r} \cdot T; \quad j^i = 0; \quad i = 1, 2, 3. \quad (19)$$

The fact that the spatial part of the magnetic current, \vec{j} , has to vanish follows from the Yang-Mills equations themselves, since $j^i = D_j \tilde{F}^{ji} + D_0 \tilde{F}^{0i} = 0$, because $\tilde{F}^{ji} = 0$, and $D_0 \tilde{F}^{0i} = 0$, since $A_0 = 0$, and the solution is static. We have then established the result given in (1), for any value of the complex parameter α appearing in the integral equations. By expanding the integral equations in powers of α , we observe that Wu-Yang solution satisfies in fact an infinity of integral equations.

We now turn to the analysis of the differential Yang-Mills equation using distribution theory [13]. The magnetic field for the Wu-Yang solution (2) is given by

$$\vec{B} = -\frac{\hat{r}}{r^2} \hat{r} \cdot T. \quad (20)$$

Since it is singular at the origin we shall analyze the equation (1) by evaluating the ordinary divergent of \vec{B} using distribution theory, i.e.

$$\int d^3\vec{r} \vec{\nabla} \cdot \vec{B} \Phi = - \int d^3\vec{r} \vec{B} \cdot \vec{\nabla} \Phi \quad (21)$$

where the test functions Φ are C^∞ functions that vanish outside a given compact region around the origin. Using spherical coordinates ($x^1 = r \sin \theta \cos \varphi$, $x^2 = r \sin \theta \sin \varphi$, and $x^3 = r \cos \theta$), we have that only the radial part of $\vec{\nabla} \Phi$ contributes to the r.h.s. of (21), and so

$$\int d^3\vec{r} \vec{\nabla} \cdot \vec{B} \Phi = \int d\theta d\varphi \sin \theta \hat{r} \cdot T [\Phi(\infty, \theta, \varphi) - \Phi(0, \theta, \varphi)]. \quad (22)$$

But $\Phi(\infty, \theta, \varphi) = 0$, since Φ vanishes outside the compact region, and $\Phi(0, \theta, \varphi) = \Phi(\vec{0})$. In addition $\hat{r} \cdot \vec{T} = \sin \theta (\cos \varphi T_1 + \sin \varphi T_2) + \cos \theta T_3$, and so

$$\int d^3\vec{r} \vec{\nabla} \cdot \vec{B} \Phi = -\Phi(\vec{0}) \int d\theta d\varphi \sin \theta \hat{r} \cdot \vec{T} = 0. \quad (23)$$

According to distribution theory one has to find a distribution u such that

$$\int d^3\vec{r} u \Phi = \int dV \vec{\nabla} \cdot \vec{B} \Phi = 0 \quad (24)$$

which leads to the result $\vec{\nabla} \cdot \vec{B} = u$. Obviously the trivial distribution $u = 0$ does the job. However, it is not the only one. Consider distributions of the type $u = C \frac{\hat{r} \cdot \vec{T}}{r^2} \delta(r)$, where C is a constant. Then one gets

$$\int d^3\vec{r} u \Phi = C \int dr d\theta d\varphi r^2 \sin \theta \frac{\hat{r} \cdot \vec{T}}{r^2} \delta(r) \Phi = C \Phi(\vec{0}) \int d\theta d\varphi \sin \theta \hat{r} \cdot \vec{T} = 0 \quad (25)$$

where in the last equality we have used the same reasoning as in (23). Therefore, one can take

$$\vec{\nabla} \cdot \vec{B} = C \frac{\hat{r} \cdot \vec{T}}{r^2} \delta(r). \quad (26)$$

Note that such analysis does not fix the value of the constant C . However, the integral Yang-Mills equations do, as we have seen above. So, to make the two results compatible we shall take $C = 1/e$. Since $[A_i, B_i] = 0$, one gets the result (1). The relation (24) implies that the differential Bianchi identity of Yang-Mills theory is satisfied in the distribution theory sense. However, there is a whole class of non-vanishing distributions u that are compatible with that fact, and so allows, in a very subtle way, the compatibility with the integral equations for the Yang-Mills theory.

Note that the gauge potential A_i given in (2) is also singular at the origin, and one could wonder if the evaluation of the curl of A_i , to obtain B_i , leads to singular terms like a Dirac string. However using distribution theory to evaluate the curl as $\int d^3\vec{r} \vec{\nabla} \wedge \vec{A} \Phi = -\int d^3\vec{r} \vec{A} \wedge \vec{\nabla} \Phi$, one gets that $\vec{\nabla} \wedge \vec{A} = \frac{2}{e} \frac{\hat{r}}{r^2} \hat{r} \cdot \vec{T}$, and so there are no singularities besides $1/r^2$.

For the Wu-Yang monopole solution (2) the electric current J^μ , given in (8), has to vanish. Indeed, the time component J^0 vanishes because the solution is static, $A_0 = 0$, and $F_{0i} = 0$. For the space components J^i one has to take care in the evaluation of $\partial_j F^{ji}$, because of the singularity at the origin. However, using distribution theory as we did above one can show that $\partial_j F_{ji} = \frac{1}{e} \varepsilon_{ijk} \frac{x^j}{r^4} T_k$, and so there is only the usual singularity at the origin $1/r^3$, and nothing else. Then one obtain that J^i has to vanish due to the differential Yang-Mills equations.

Another point one could rise is the application of the generalized non-abelian Stokes theorem for field configurations that present singularities, like the Wu-Yang monopole. We would

like to stress that the only assumption needed in the proof of that theorem is the commutability of the space-time derivatives. However that property always holds true if one adopts the definition of derivatives, close to the singularities, in the distribution theory sense. In addition, the evaluation of the Wilson operator for loops that pass through the Wu-Yang monopole singularity presents no problem as explained in the appendix of [6].

As a final comment we would like to point out that the integral Yang-Mills equations lead to conserved charges that are invariant under arbitrary gauge transformations as explained in [6, 7, 14]. Those charges are the eigenvalues of the operator $V(\partial\Omega_\infty) = U(\Omega_\infty)$, where Ω_∞ is the whole space sub-manifold of the space-time under consideration. Since, those operators involve two arbitrary parameters α and β (see (9)) one can expand the operators V and U in powers of those parameters and get in principle an infinite number of charges. In the examples calculated so far the higher charges are powers of the first ones [6, 7, 14]. The conserved charges are the eigenvalues of the magnetic charge operator which in the case of the Wu-Yang monopole becomes

$$Q_M = \int_{\partial\Omega_\infty} d\tau d\sigma W^{-1} F_{ij} W \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} = \frac{4\pi}{e} T_R. \quad (27)$$

That should be contrasted to the usual Noether magnetic charge given in the literature, and that vanishes for the Wu-Yang monopole (see [1, 15])

$$\tilde{Q}_M = \int_{\partial\Omega_\infty} d\tau d\sigma F_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau} = 0. \quad (28)$$

The vanishing of the charge (28) is because F_{ij} is proportional to $\hat{r} \cdot T$ (see (2)) and the integral in the angles vanish for the same reason it does in (23). On the other hand, the conjugation of F_{ij} with the Wilson line operator W in (27), not only makes the charge gauge invariant for general (large) gauge transformations [6, 7], but prevents it from being zero. Therefore we have established that the Wu-Yang monopole solution does possess a dynamically conserved non-vanishing magnetic charge associated to it.

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